Restrictions on Magnetic Charge from Quantized Angular Momentum

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Abstract

Using the result that an electric charge - magnetic charge system carries an internal field angular momentum of $eg/4\pi$ we arrive at two restrictions on magnetic monopoles via the requirement of angular momentum quantization and/or conservation. First we show that magnetic charge should scale in the opposite way from electric charge. Second we show that *free*, *unconfined* monopoles seem to be inconsistent with quantized angular momentum when one considers a magnetic charge in the vicinity of more than one electric charge.

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I. RUNNING OF THE MAGNETIC COUPLING

One of the most unusal results of including magnetic charges in electromagnetism is that an isolated electric charge - magnetic charge system carries an angular momentum in the electric and magnetic fields of the system. If the electric charge has a magnitude e and the magnetic charge a magnitude g the angular momentum carried in the $\mathbf{E}$ and $\mathbf{B}$ fields is

$$\mathbf{L}_{fields} = \frac{eg}{4\pi}\hat{\mathbf{r}} \tag{1}$$

where $\hat{\mathbf{r}}$ is the unit vector pointing from the electric charge to the magnetic charge [1]. This result, along with the quantum mechanical requirement that all angular momentum come in integer units of $\hbar/2$ is the best formulation independent way of arriving at the Dirac quantization condition that $eg/4\pi = n(\hbar/2)$ $n = 1, 2, 3, ...$ [2]. (It is formulation independent in that it only requires the assumption of the Coulombic form of the $\mathbf{E}$ and $\mathbf{B}$ fields surrounding the electric and magnetic charges. It is not necessary to assume either the string approach of Dirac [3] or the fiber-bundle approach of Wu and Yang [4]). Using Eq. (1) and the conservation and quantization of angular momentum it is possible to determine how the magnetic coupling g scales with energy.

The running with energy of the coupling of a theory is determined by the beta function which is defined as

$$\beta_\lambda = \mu\frac{\partial\lambda}{\partial\mu} \tag{2}$$

where λ is the coupling and μ is an arbitrary energy scale. The electric coupling e, and the magnetic coupling g, as with any other coupling, should have a dependence on the energy scale μ. The usual way of finding $e(\mu)$, for example, is to calculate the beta function perturbatively to the desired order in perturbation theory, plug this into the left hand side of Eq. (2), and then solve the resulting differential equation for $e(\mu)$. Since the magnetic coupling $g(\mu)$ is nonperturbatively large (from the Dirac quantization condition and e being perturbatively small) it is difficult to calculate the magnetic beta function, β_g, in the usual

way. For an isolated electric charge - magnetic charge system the quantity $e(\mu)g(\mu)/4\pi$ must be a constant since otherwise angular momentum conservation would be violated. Additionally $e(\mu)$ and $g(\mu)$ must scale in a related way so that the field angular momentum is always equal to some integer multiple of $\hbar/2$. Since $e(\mu)$ and $g(\mu)$ run in a continuous way the only way to satisfy this requirement of quantized angular momentum is for $e(\mu)g(\mu)/4\pi$ to remain equal to whatever integer multiple of $\hbar/2$ it was equal to at the reference energy scale μ_0. Thus taking the Dirac condition $e(\mu)g(\mu)/4\pi = n(\hbar/2)$ differentiating both sides by μ and multiplying by $4\pi\mu$ one gets

$$e\left(\mu\frac{\partial g}{\partial\mu}\right) + g\left(\mu\frac{\partial e}{\partial\mu}\right) = 0 \tag{3}$$

which with the help of Eq. (2) can be written as

$$\beta_g = -\frac{g}{e}\beta_e \tag{4}$$

The subscripts indicate the magnetic or electric coupling. The minus sign in Eq. (4) shows that the magnetic coupling and electric coupling should run in opposite ways. Moreover Eq. (4) should be valid to any order of perturbation theory since it just depends on the requirement that angular momentum be conserved and quantized. In order to make any detailed statements about the scaling of $e(\mu)$ or $g(\mu)$ one must be able to calculate either β_g or β_e. Taking as an example spinor QED where $\beta_e = e^3/12\pi^2$, and using the Dirac quantization condition to replace e with g via $e = 2\pi/g$ (where we have specialized to the $n = 1$ case and set $\hbar = 1$) it is found that Eq. (4) becomes $\beta_g = \frac{-1}{3g}$. Solving this for $g(\mu)$ yields

$$g^2(\mu) = g^2(\mu_0) - \frac{2}{3}\ln\left(\frac{\mu}{\mu_0}\right) \tag{5}$$

where μ_0 is the reference energy scale. From Eq. (5) it is easy to see that the magnetic coupling decreases with energy as opposed to the electric coupling, $e(\mu)$, which increases with energy. The decrease is logarithmically slow so that the magnetic coupling will still remain enormous out to energy scales beyond which the perturbative calculation of β_e can

be trusted, and far out of reach of any current or planned accelerators. This example, or a more realistic extension using β_e of the Standard Model, depend on the implicit assumption that monopoles can be ignored in the perturbative calculation of β_e. Recently it has been shown that experimentally measured Standard Model parameters do not show any sign of the effect of virtual monopoles up to roughly 1 TeV [5]. The harder and more interesting question of how to handle the situation when virtual monopoles do start to effect β_e is left unanswered by the above results, but it is still somewhat interesting that the scaling of the magnetic coupling can be determined in a region where its strength is nonperturbatively large.

Several other authors [6] [7] have investigated quantum field theories with magnetic charge, and have discussed the renormalization of the Dirac quantization condition as well as the running of the magnetic coupling. In Ref. [6] it is claimed that magnetic and electric couplings should scale in the same way, while Ref. [7] comes to the different conclusion that they should scale in the opposite way. Our results agree with those of Ref. [7], but our argument is different. Rather than using a perturbative, Feynman diagrammatic approach we arrive at our results based on angular momentum conservation and quantization. Thus, although our result is the same as that of Ref. [7], our arguments should remain valid to all orders of perturbation theory.

II. MONOPOLE AND TWO ELECTRIC CHARGES

From Eq. (1) one sees that an electric charge - magnetic charge system carries an internal angular momentum due to the **E** and **B** fields of the particles, which has a magnitude $eg/4\pi$ and points from the electric charge toward the magnetic charge. Now we will consider what happens when the monopole is in the presence of two electric charges. For definiteness we place a positive electric charge, $+e$, at the origin, a negative electric charge, $-e$, along the z-axis a distance $d > 1$ away from the origin, and we place the magnetic charge, g, at an arbitrary point on the unit circle in the xz plane so that its position is given by the polar

angle θ. The angular momentum of the $+eg$ part of this system points from $+e$ to g and is

$$\mathbf{L}_1 = \frac{eg}{4\pi} \left(\cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{z}} \right) \tag{6}$$

The angular momentum of the $-eg$ part of the system points from g to $-e$ and is

$$\mathbf{L}_2 = \frac{eg}{4\pi} \left(\frac{-\cos\theta}{\sqrt{d^2 + 1 - 2d\sin\theta}} \hat{\mathbf{x}} + \frac{d - \sin\theta}{\sqrt{d^2 + 1 - 2d\sin\theta}} \hat{\mathbf{z}} \right) \tag{7}$$

The total angular momentum of this system of two electric charges and one magnetic charge is then simply the sum of Eqs. (6) and (7)

$$\mathbf{L}_{tot} = \mathbf{L}_1 + \mathbf{L}_2 \tag{8}$$

Now the magnitude of $\mathbf{L}_{tot}$ varies continuously from $2(eg/4\pi)$ at $\theta = \pi/2$, to 0 at $\theta = 3\pi/2$. Thus, classically, any value for $|\mathbf{L}_{tot}|$ between $eg/2\pi$ and 0 is possible depending on the choice of θ. If one requires that a particular value of $|\mathbf{L}_{tot}|$, for a particular θ, equal some integer multiple of $\hbar/2$ then all the continuously connected values of $|\mathbf{L}_{tot}|$ at neighboring values of θ will not equal an integer multiple of $\hbar/2$, in violation of the requirement that angular momentum be quantized. Further for $\theta = 0$ or π one finds that

$$|\mathbf{L}_{tot}| = \frac{eg}{4\pi} \sqrt{2 - \frac{2}{\sqrt{d^2 - 1}}} \tag{9}$$

so that $|\mathbf{L}_{tot}|$ also has some dependence on d as well as on θ. The only way for the total angular momentum of this system to be quantized would be for the initial positions of the particles (as determined by θ and d) to take only certain discrete values (*i.e.* for the positions to be quantized). Since there is no apparent mechanism that requires any such positional "quantization" of the initial configuration of our system we conclude that such a configuration (or more precisely the free monopole in the configuration) seems to be inconsistent with the quantization of angular momentum. One may worry about the use of the classical Coulomb fields for the electric and magnetic charges in the above development (especially in deriving Eq. (1)). However, the field angular momentum given in Eq. (1) is independent of the distance between the charges so that we may take all the charges to

be arbitrarily far away from one another (*e.g.* have the magnetic charge placed on a circle of radius R instead of unit radius such that $d \gg R \gg 1$) without affecting the result of Eq. (1). Under these conditions one is certainly justified in using the Coulomb form for the electric and magnetic fields of the particles, and ignoring any quantum corrections.

Since electric charges are known to exist, the above argument seems to imply that free magnetic charges do not exist. Despite this it may still be possible that monopoles exist, if they are permanently confined much in the same way that color charged quarks are postulated to be permanently confined. This seems reasonable in light of the fact that, from the Dirac condition, the monopole coupling is an order of magnitude, or more, larger than the equivalent QCD coupling which is thought to confine quarks. In addition Wilson's [8] original lattice gauge theory paper on confinement uses a U(1) gauge field to argue for confining behaviour, so that it is not the Abelian (QED) versus non-Abelian (QCD) character of an interaction that is of chief importance but the strength of the coupling. It may be argued that Wilson's results can not be applied to magnetic charge since it is the electric charge which is the gauge charge and is directly coupled to the U(1) gauge boson. However, as pointed out in Ref. [9], it is just as easy to make the magnetic charge the gauge charge, which is directly coupled to the U(1) gauge boson, by introducing a four vector potential, $C_\mu = (\phi_m, \mathbf{C})$, which is dual to the usual four vector potential A_μ. Then by defining a new field strength tensor and its dual as

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu \qquad \mathcal{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta} \tag{10}$$

one can obtain the $\mathbf{E}$ and $\mathbf{B}$ fields as

$$E_i = -\mathcal{G}^{i0} \qquad B_i = G^{i0} \tag{11}$$

(In three vector notation this becomes $\mathbf{E} = -\nabla \times \mathbf{C}$ and $\mathbf{B} = -\nabla\phi_m - (1/c)(\partial \mathbf{C}/\partial t)$). In this dual formulation it is the magnetic charge with is the gauge charge and the electric charge which finds itself on the end of a Dirac string. Thus Wilson's arguments for confinement should apply equally well to magnetic charge in this dual formulation of electrodynamics.

Although in principle either formulation of electrodynamics (in terms of A_μ or C_μ) is equally satisfactory, in practice it much easier to deal with whatever charge is directly coupled to the U(1) gauge boson. Since electric charges are known to exist while magnetic charges have never been seen experimentally it is more practical to do electrodynamics with electric charge directly coupled to the U(1) gauge boson. (An interesting possibility in electromagnetism with both electric and magnetic charges is to use both A_μ and C_μ, and have one of the extra "photons" (*i.e.* A_μ or C_μ) hidden through some form of symmetry breaking [12]).

III. CONCLUSIONS

Using the result that a magnetic charge placed in the field of one or more electric charges produces a field angular momentum, and that angular momentum must be conserved and quantized, we have given two restrictions on the behaviour of monopoles. First Eq. (4) shows that the magnetic coupling will scale in the opposite way from the electric coupling. Although the running of the magnetic coupling, as given in the example of spinor QED, is logarithmically slow, so that the magnetic coupling will still be enourmous out to any energy scale that may be reasonably reached at any future accelerator, it is interesting that one can say anything about the scaling of a coupling that is nonperturbatively large. This result is not new [7], but we have arrived at it using only angular momentum conservation and quantization so that our result, as given in Eq. (4), should be good to any order. Second, by considering the field angular momentum of a system of two electric charges and one monopole we have found that it is not possible to have the field angular momentum quantized to some integer multiple of $\hbar/2$ for every possible initial position of this system (*i.e.* angular momentum can not be quantized since the initial positions of the charges need not be "quantized"). Since the quantization of angular momentum is a fundamental requirement of quantum mechanics this appears to imply that free, unconfined monopoles are incompatiable with quantum mechanics and therefore do not exist. A possible evasion of this argument is that even if there are no free monopoles they may exist and be permanently

confined, in the same way quarks are thought to be permanently confined. (This confinement hypothesis for magnetic charge also fits in with the scaling of the magnetic coupling discussed in the first section). The dual formulation of QED [9], where magnetic charge is the gauge charge, combined with the lattice gauge theory confinement arguments of Wilson [8] also help to give strength to this confinement hypothesis of magnetic charge. This would also give an explanation as to why free monopoles have evaded experimental detection, just as free quarks have evaded detection.

Finally one may ask how the above arguments apply to 't Hooft-Polyakov [10] monopoles. It would appear that since our arguments depend on the electric charge - magnetic charge system carrying a field angular momentum that they should apply to these topological magnetic charges as well. It has been shown that the composite system of a 't Hooft-Polyakov monopole and a non-Abelian test particle also carries a field angular momentum [11]. However in this case things may be a bit more delicate since it is not clear that one can use the principle of superposition as freely as in the Abelian case. For example, if the Abelian monopole were replaced by a 't Hooft-Polyakov monopole, and the two electric charges by non-Abelian charges, and if the superposition were valid, then a conflict with quantized angular momentum would also arise in this non-Abelian system. However non-Abelian gauge theories are non-linear so the superposition principle is not valid. Without further analysis the best that can be done is to speculate that since the field angular momentum is again independent of the distance between the charges that all the charges could again be separated by arbitrarily large distances so that superposition would be approximately valid.

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